Poisson

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HURRICANE MICHAEL
11:00 AM CDT
LAT: 29.6°N LON: 85.8°W
35 MI SW OF MEXICO BEACH FLORIDA
WINDS: 150 MPH
PRESSURE: 923 mb
MOVING: NNE at 14 MP
Birmingham
Macon
Jacksonville
Panama City
Probability for Extreme Weather?

Review

Binomial Random Variable

- Consider ***n* independent trials** of an experiment with success probability *p*.
  - X is **number of successes** in *n* trials
  - X is a **Binomial** Random Variable:

- Examples
  - # of heads in *n* coin flips
  - # of 1's in randomly generated length *n* bit string
  - # of disk drives crashed in 1000 computer cluster
    - Assuming disks crash independently

Our random variable

Num trials

Probability of success on each trial

$$X \sim \mathrm{Bin}(n, p)$$

Is distributed as a

Binomial

With these parameters

If X is a binomial with parameters n and p

Probability Mass Function for a Binomial

$$P(X = k) = \binom{n}{k} p^k (1-p)^{n-k}$$

Probability that our variable takes on the value k

Bernoulli vs Binomial

$$X \sim \mathrm{Bern}(p)$$

$$X \in \{0, 1\}$$

Bernoulli is a type of RV that can take on two values, 1 (for success) with probability *p* and 0 (for failure) with probability (1- *p*)

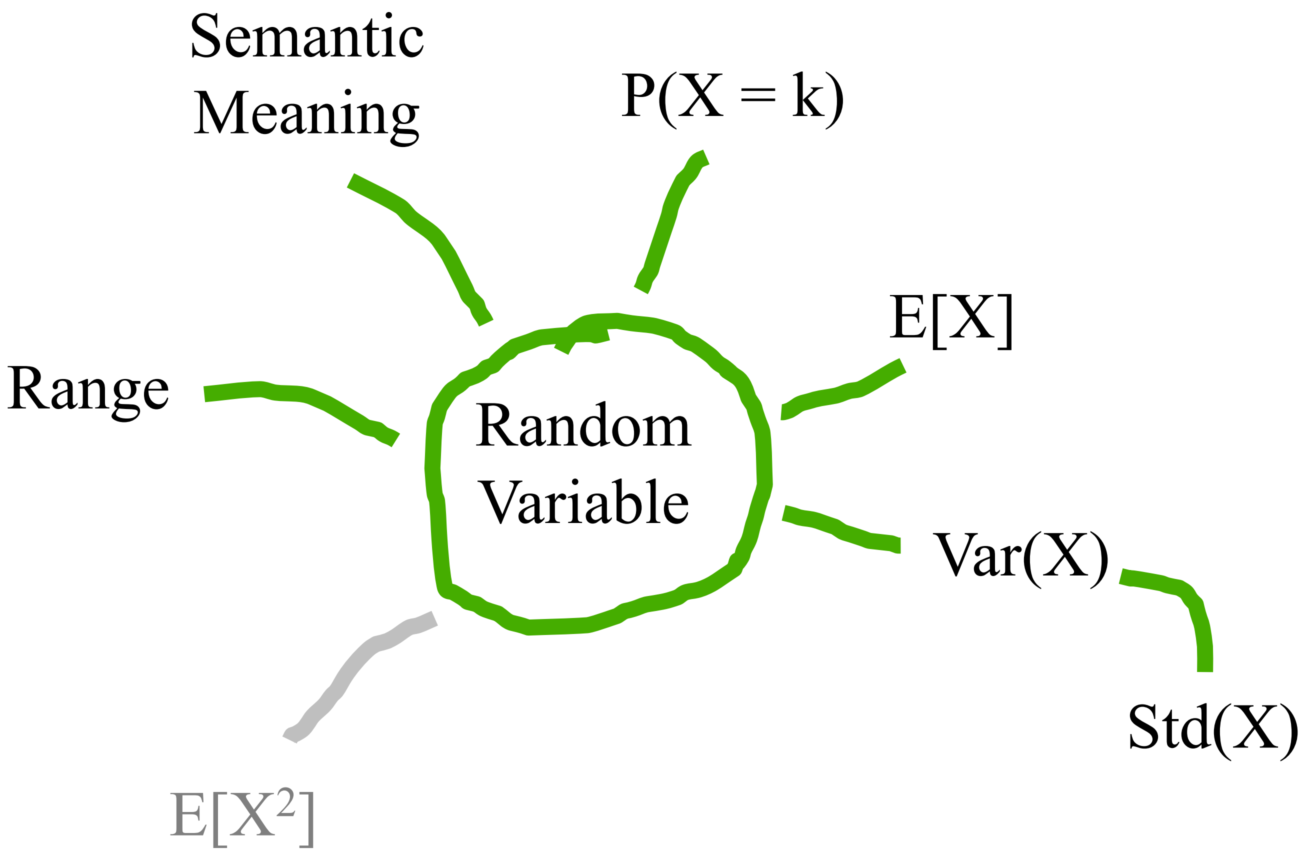
$$Y \sim \mathrm{Bin}(n, p)$$

$$Y = \sum_{i=1}^{n} X_i$$

$$\text{s.t. } X_i \sim \mathrm{Bern}(p)$$

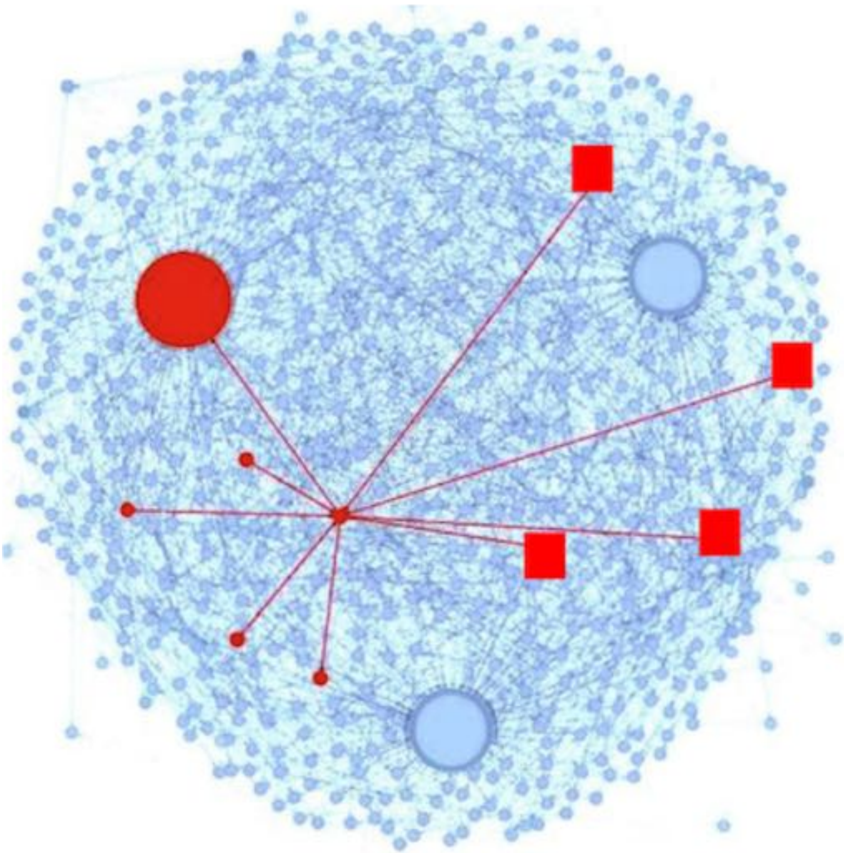
Binomial is the sum of *n* Bernoullis

Fundamental Properties



Is Peer Grading Accurate Enough?

Looking ahead



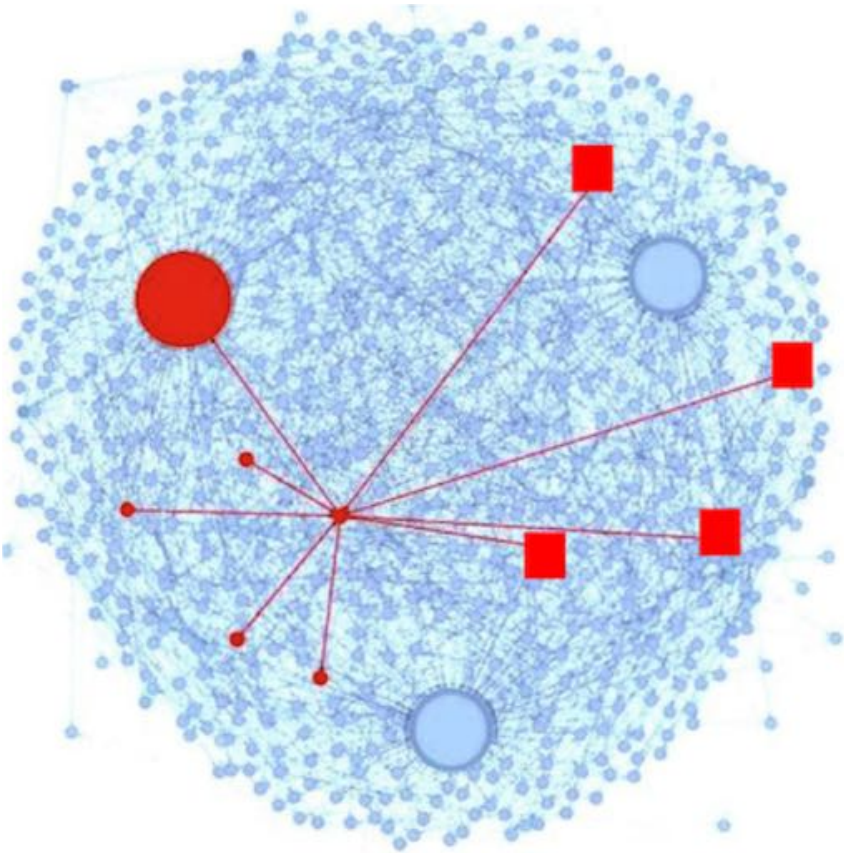
Peer Grading on Coursera HCI.

31,067 peer grades for 3,607 students.

Tuned Models of Peer Assessment. C Piech, J Huang, A Ng, D Koller

Is Peer Grading Accurate Enough?

Looking ahead



1. Defined random variables for:

- True grade (s_i) for assignment i
- Observed (z_i^j) score for assign i
- Bias (b_j) for each grader j
- Variance (r_j) for each grader j

2. Designed a probabilistic model that defined the distributions for all random variables

Problem param

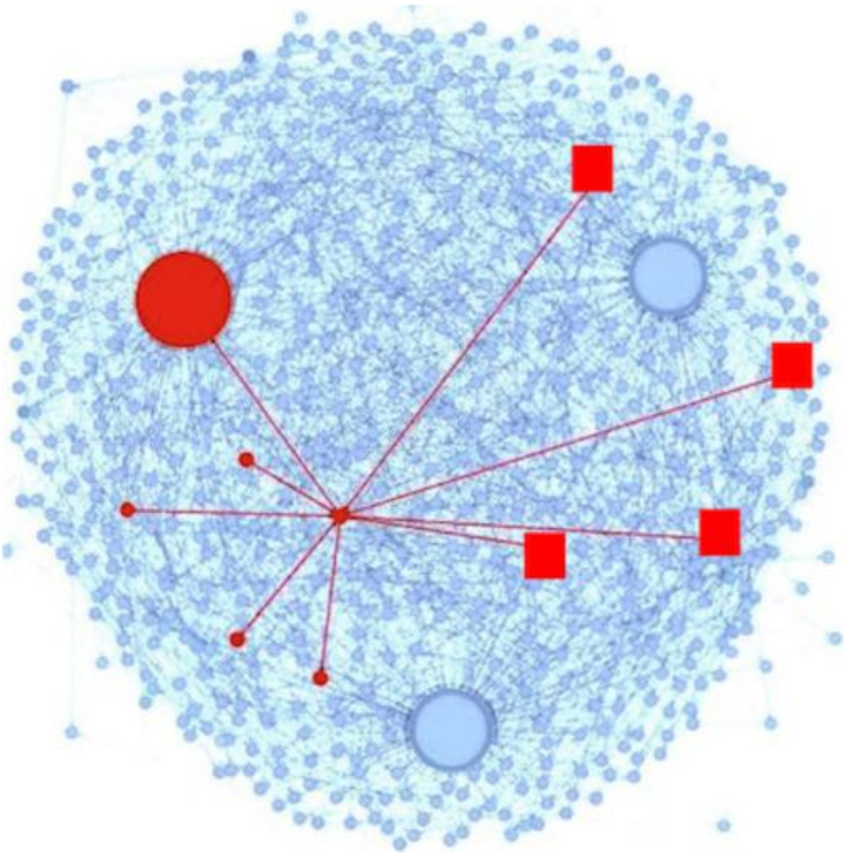
$$s_i \sim \text{Bin}(\text{points}, \theta)$$

$$z_i^j \sim \mathcal{N}(\mu = s_i + b_j, \sigma = \sqrt{r_j})$$

Tuned Models of Peer Assessment. C Piech, J Huang, A Ng, D Koller

Is Peer Grading Accurate Enough?

Looking ahead



1. Defined random variables for:
- True grade (s_i) for assignment i
- Observed (z_i^j) score for assign i
- Bias (b_j) for each grader j
- Variance (r_j) for each grader j

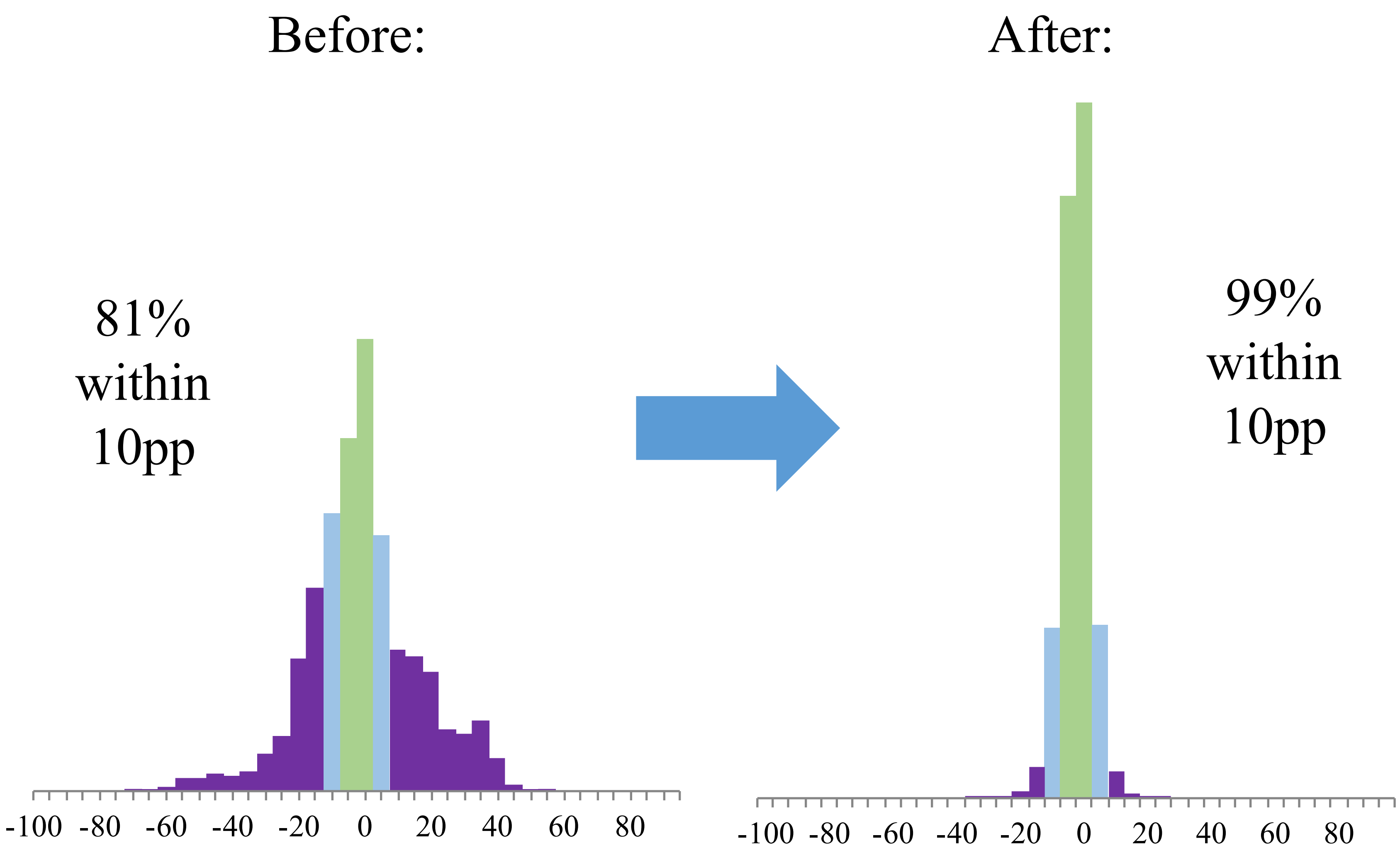
2. Designed a probabilistic model that defined the distributions for all random variables

3. Found the variable assignments that maximized the probability of our observed data

Inference or Machine Learning

Tuned Models of Peer Assessment. C Piech, J Huang, A Ng, D Koller

Yes, With Probabilistic Modelling



Tuned Models of Peer Assessment. C Piech, J Huang, A Ng, D Koller

End Review

Good to know

Natural Exponent def:

$$\lim_{n\to\infty}\left(1-\frac{\lambda}{n}\right)^n = e^{-\lambda}$$

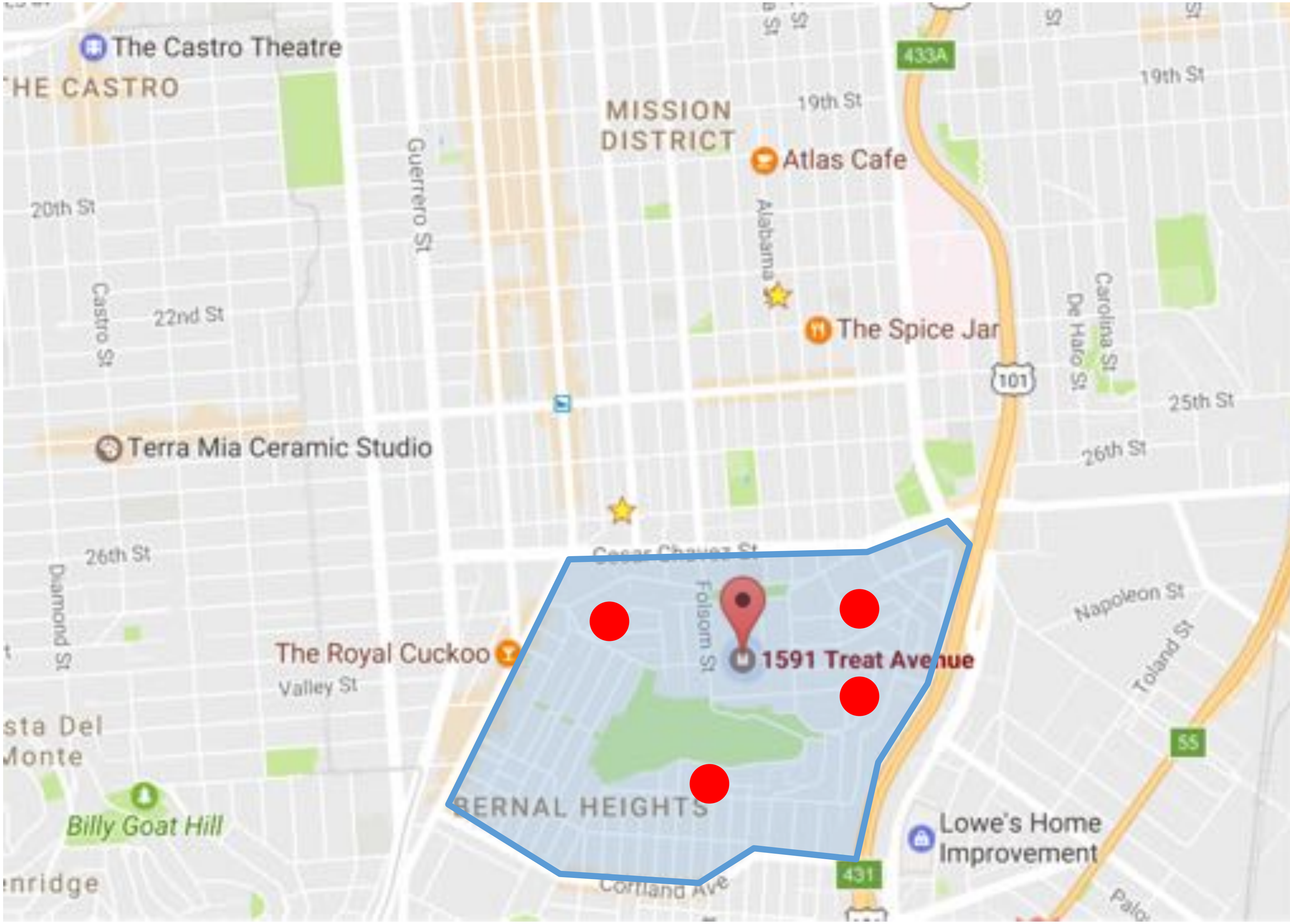
Jacob Bernoulli

https://en.wikipedia.org/wiki/E_(mathematical_constant)

Algorithmic Ride Sharing

Probability of *k* requests from this area in the next 1 min

Probability of *k* requests from this area in the next 1 min



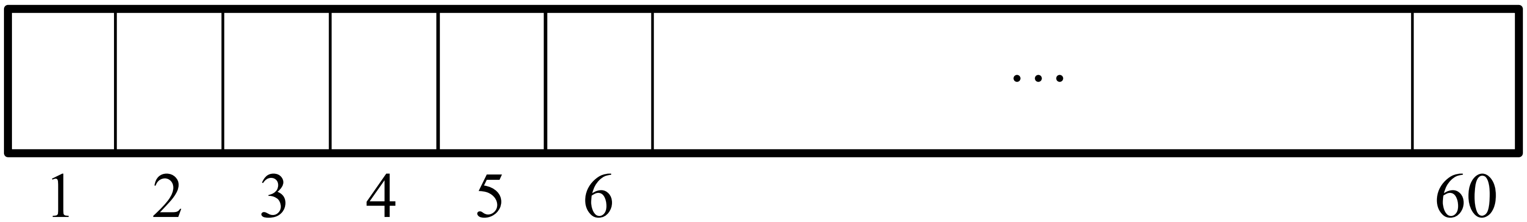
Probability of *k* requests from this area in the next 1 min

On average λ = 5 requests per minute

Probability of *k* requests from this area in the next 1 min

On average $\lambda = 5$ requests per minute

We can break the next minute down into seconds



Probability of *k* requests from this area in the next 1 min

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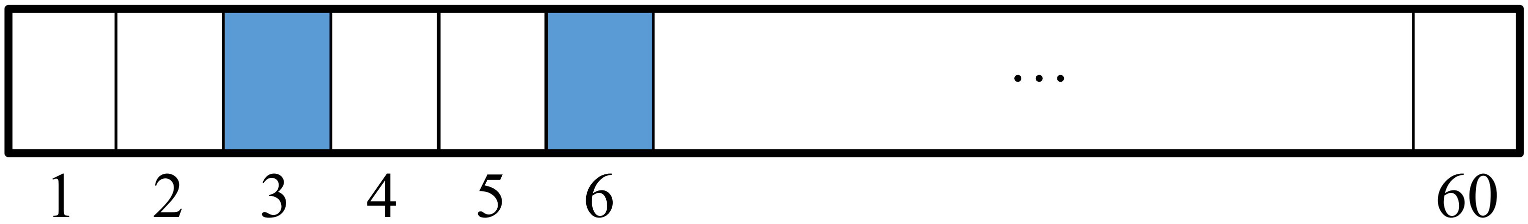


At each second either get a request or you don't.

Probability of *k* requests from this area in the next 1 min

On average λ = 5 requests per minute

We can break the next minute down into seconds

At each second either get a request or you don't.
Let X = Number of requests in the minute

$$X \sim \text{Bin}(n = 60, p = 5/60)$$

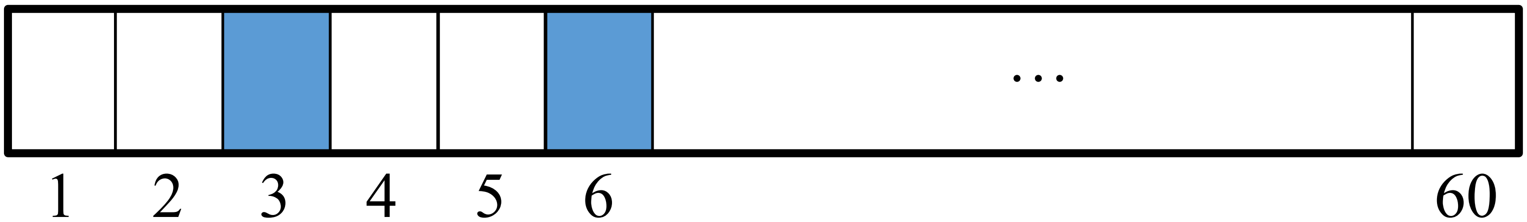
$$P(X = k) = \binom{n}{k}(p)^k(1-p)^{n-k}$$

$$P(X = 3) = \binom{60}{3}(5/60)^3(1-5/60)^{57}$$

Probability of *k* requests from this area in the next 1 min

On average λ = 5 requests per minute

We can break the next minute down into seconds



At each second either get a request or you don't.

Let X = Number of requests in the minute

$$X \sim \text{Bin}(n = 60, p = 5/60)$$

$$P(X = k) = \binom{n}{k} (p)^k (1 - p)^{n-k}$$

But what if there are two requests in the same second?

Probability of *k* requests from this area in the next 1 min

On average $\lambda = 5$ requests per minute

We can break that next minute down into ***milli***-seconds

At each ***milli***-second either get a request or you don't.
Let X = Number of requests in the minute

But what if there are two requests in the same second?

Probability of *k* requests from this area in the next 1 min

On average λ = 5 requests per minute

We can break that next minute down into ***milli***-seconds

At each ***milli***-second either get a request or you don't.
Let X = Number of requests in the minute

$$X \sim \text{Bin}(n = 60000, p = \lambda/n)$$

$$P(X = k) = \binom{n}{k}(\lambda/n)^k(1 - \lambda/n)^{n-k}$$

Can we do any better than milli-seconds?

Binomial in the Limit

On average λ = 5 requests per minute

We can break that minute down into ***infinitely small*** buckets

OMG so small

1 ∞

Let X = Number of requests in the minute

$$X \sim \text{Bin}(n, p = \lambda/n)$$

$$P(X = k) = \lim_{n \to \infty} \binom{n}{k} (\lambda/n)^k (1 - \lambda/n)^{n-k}$$

Who wants to see some cool math?

Binomial in the Limit

$$P(X = k) = \lim_{n \to \infty} \binom{n}{k} (\lambda/n)^k (1 - \lambda/n)^{n-k}$$

$$= \lim_{n \to \infty} \frac{n!}{(n-k)!k!} \cdot \frac{\lambda^k}{n^k} \cdot \frac{(1 - \lambda/n)^n}{(1 - \lambda/n)^k}$$ By expanding each term

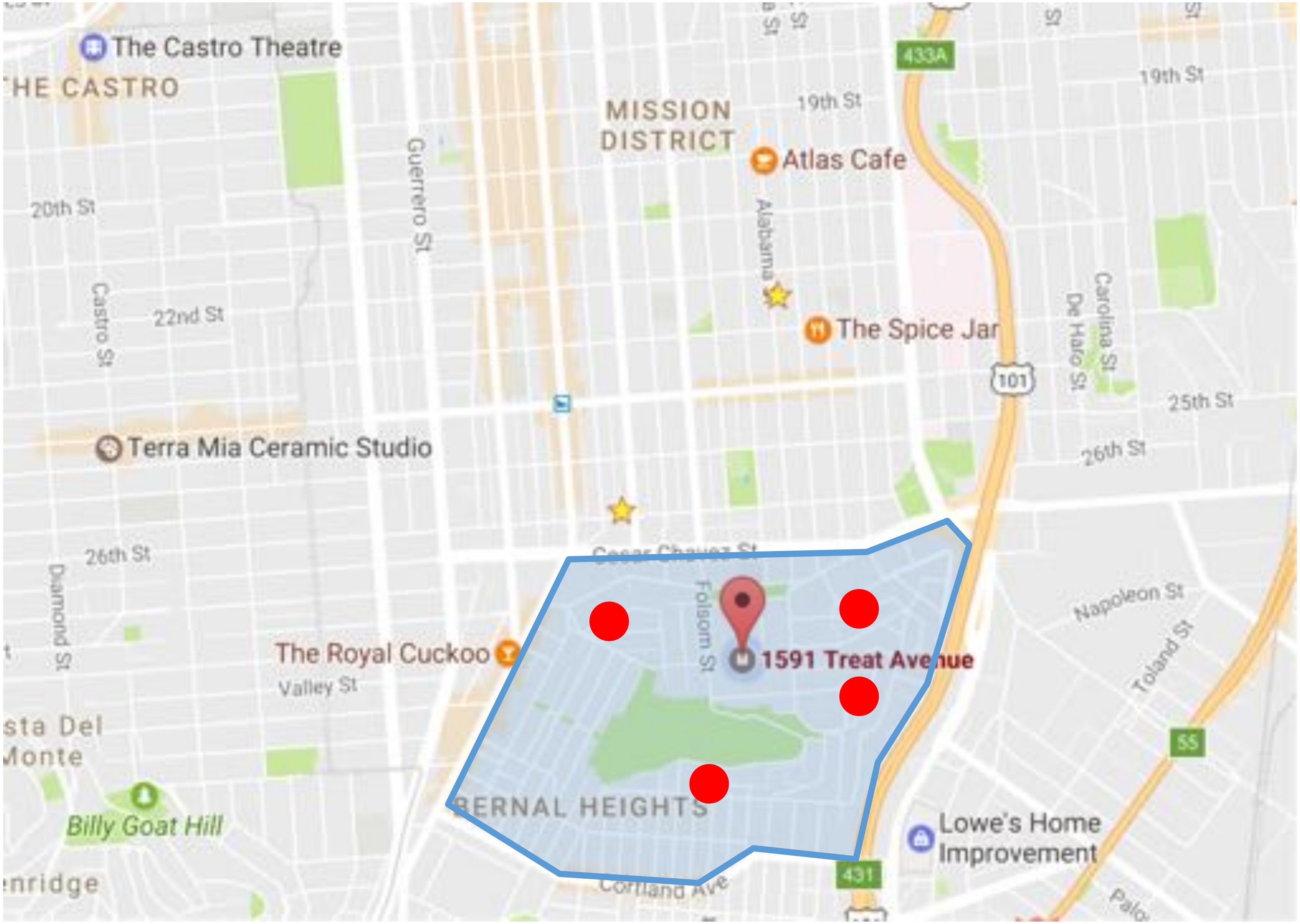
$$= \lim_{n \to \infty} \frac{n!}{(n-k)!k!} \cdot \frac{\lambda^k}{n^k} \cdot \frac{e^{-\lambda}}{1}$$ By definition of natural exp

$$= \lim_{n \to \infty} \frac{n!}{(n-k)!n^k} \cdot \frac{\lambda^k}{k!} \cdot \frac{e^{-\lambda}}{1}$$ Rearranging terms

$$= \lim_{n \to \infty} \frac{n^k}{n^k} \cdot \frac{\lambda^k}{k!} \cdot \frac{e^{-\lambda}}{1}$$ Limit analysis

$$= \frac{\lambda^k e^{-\lambda}}{k!}$$ Simplifying

Probability of *k* requests from this area in the next 1 min



Simeon-Denis Poisson

- Simeon-Denis Poisson (1781-1840) was a prolific French mathematician

- Published his first paper at 18, became professor at 21, and published over 300 papers in his life
  - He reportedly said *"Life is good for only two things, discovering mathematics and teaching mathematics."*
- I'm going with French Martin Freeman

Poisson Random Variable

- X is a **Poisson** Random Variable: the number of occurrences in a fixed interval of time.

$$X \sim \text{Poi}(\lambda)$$

  - λ is the "rate"
  - X takes on values 0, 1, 2…
  - has distribution (PMF):

$$P(X = k) = e^{-\lambda} \frac{\lambda^k}{k!}$$

Poisson Process

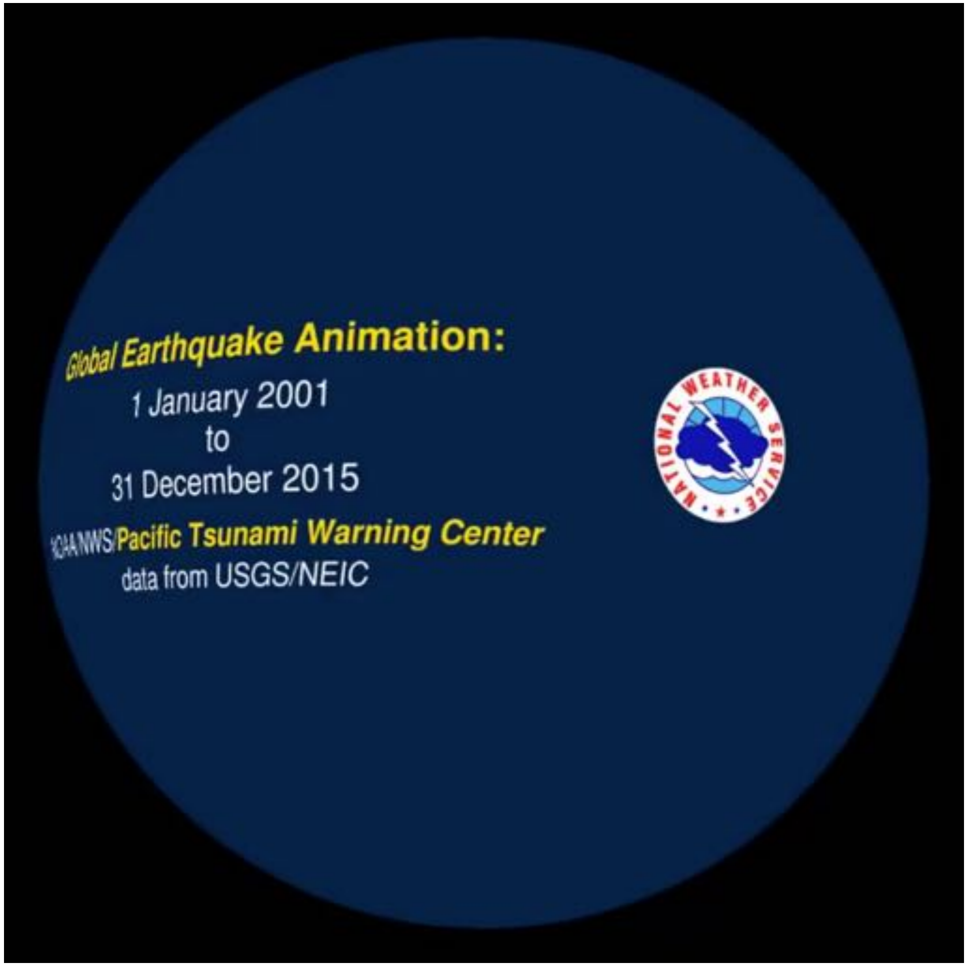
- Consider events that occur over time
  - Earthquakes, radioactive decay, hits to web server, etc.
  - Have time interval for events (1 year, 1 sec, whatever...)
  - Events arrive at rate: λ events per interval of time
- Split time interval into $n \rightarrow \infty$ sub-intervals
  - Assume at most one event per sub-interval
  - Event occurrences in sub-intervals are independent
  - With many sub-intervals, probability of event occurring in any given sub-interval is small
- $N(t)$ = # events in original time interval ~ $\text{Poi}(\lambda)$

Poisson is great when you have a rate!

Poisson is great when you have a rate and you care about # of occurrences!

Make sure that the time unit for “rate” and match the probability question

Earthquakes



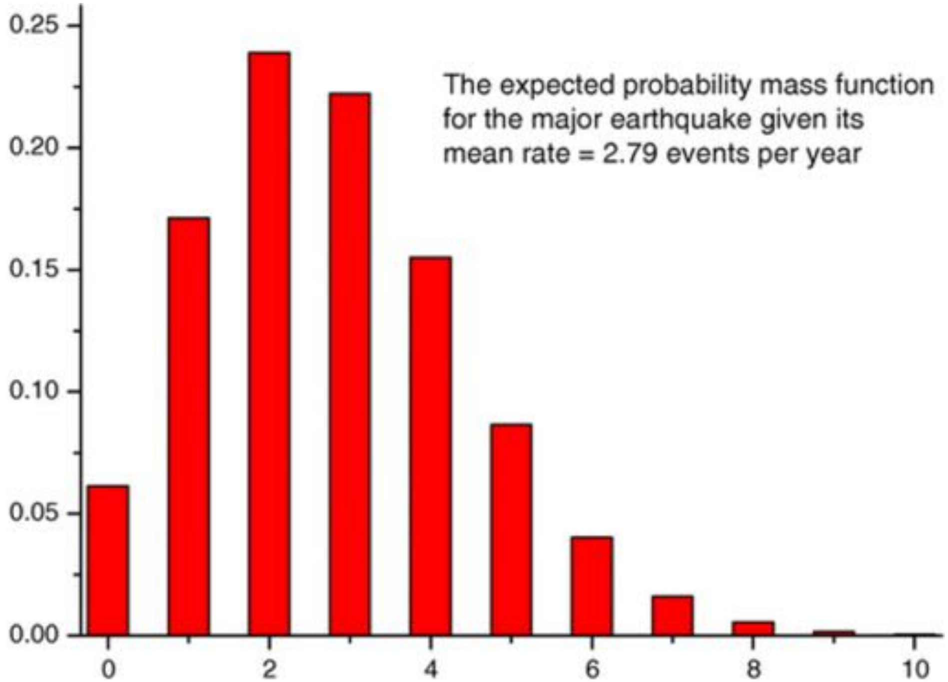
Average of 2.79 major earthquakes per year.
What is the probability of 3 major earthquakes next year?

Earthquake Probability Mass Function

Let X = number of earthquakes next year

$$X \sim \text{Poi}(2.79)$$

$P(X = x)$



Number of earthquakes (x)

$$P(X = 3) = \frac{\lambda^k e^{-\lambda}}{k!} = \frac{2.79^3 e^{-2.79}}{3!} \approx 0.23$$

Bulletin of the
Seismological Society of America

Vol. 64 October 1974 No. 5

IS THE SEQUENCE OF EARTHQUAKES IN SOUTHERN CALIFORNIA, WITH AFTERSHOCKS REMOVED, POISSONIAN?

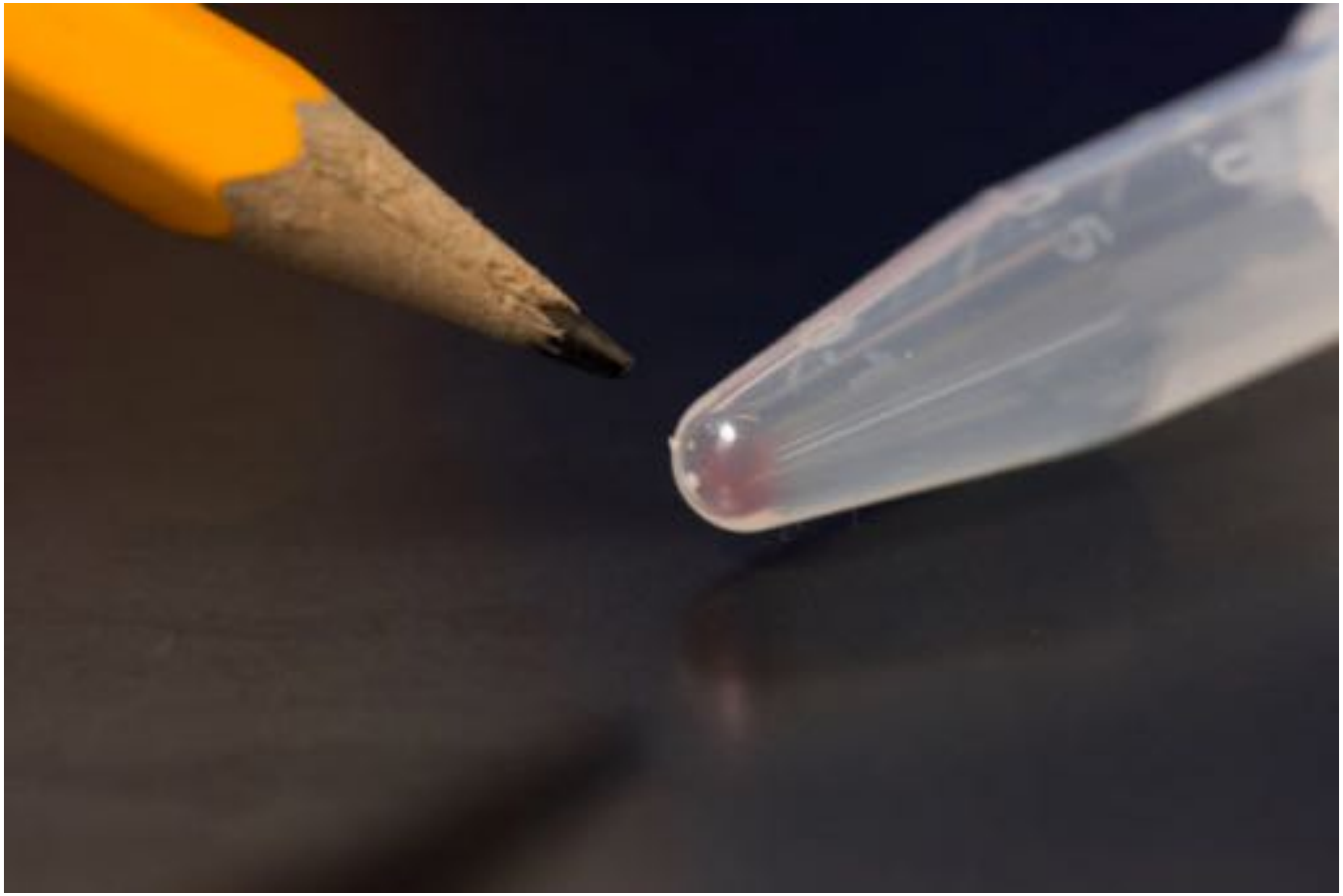
BY J. K. GARDNER and L. KNOPOFF

ABSTRACT

Yes.

Poisson can approximate a Binomial!

Storing Data on DNA



All the movies, images, emails and other digital data from more than 600 smartphones (10,000 gigabytes) can be stored in the faint pink smear of DNA at the end of this test tube.

Storing Data on DNA

- Will the DNA storage become corrupt?
  - In DNA (and real networks) store large strings
  - Length $n \approx 10^4$
  - Probability of corruption of each base pair is very small $p \approx 10^{-6}$
  - $X \sim Bin(10^4, 10^{-6})$ is unwieldy to compute

- Extreme *n* and *p* values arise in many cases
  - # bit errors in steam sent over a network
  - # of servers crashes in a day in giant data center

Storing Data on DNA

- Will the DNA storage become corrupt?
  - In DNA (and real networks) store large strings
  - Length $n \approx 10^4$
  - Probability of corruption of each base pair is very small $p \approx 10^{-6}$
  - X ~ Poi($\lambda = 10^4 * 10^{-6} = 0.01$)

$$P(X = k) = e^{-\lambda}\frac{\lambda^k}{k!}$$

$$P(X = 0) = e^{-\lambda}\frac{1}{0!}$$

$$= e^{-0.01} \approx 0.99$$

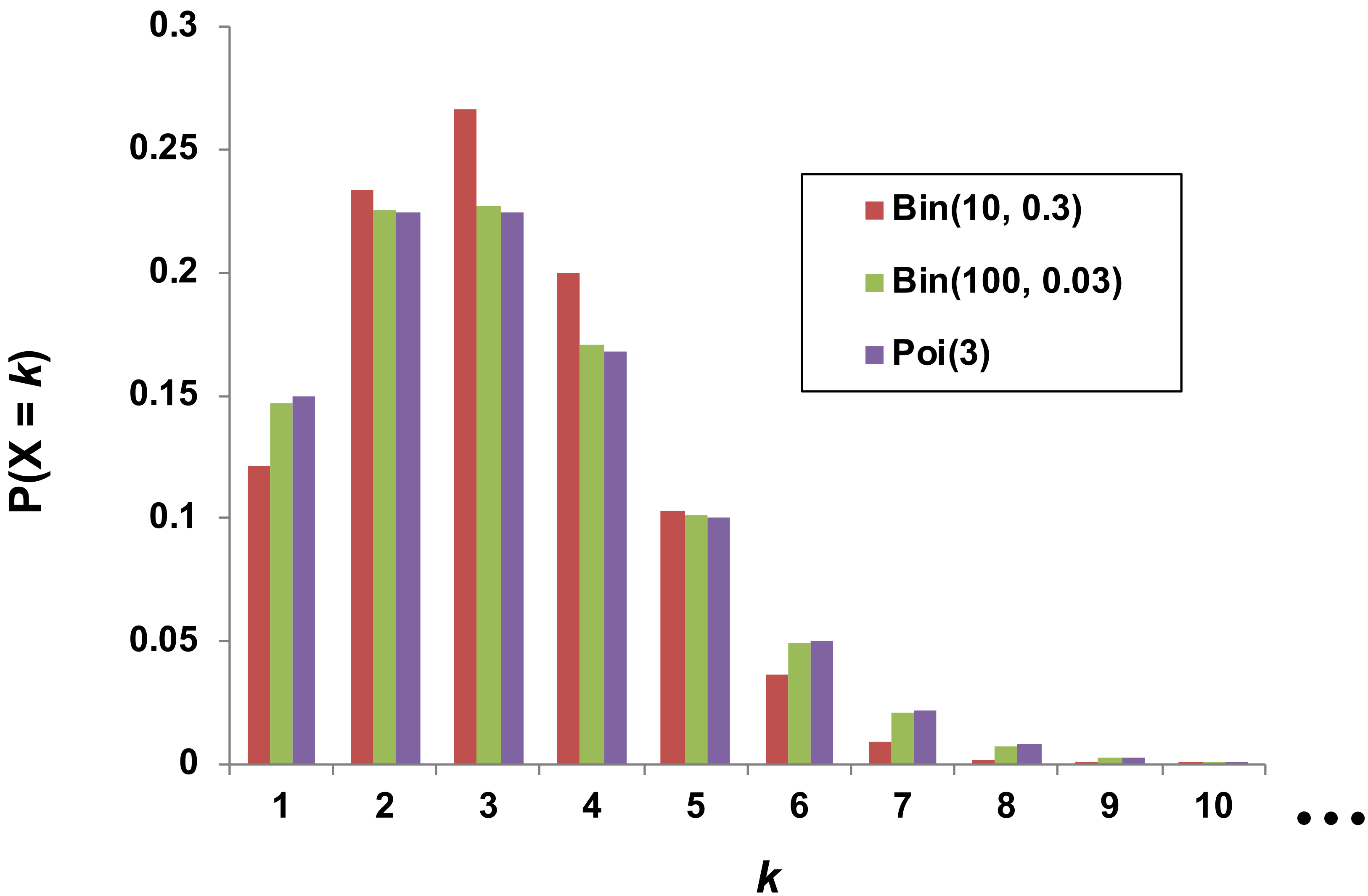
Poisson is Binomial in the Limit

- Poisson approximates Binomial where n is large, p is small, and $\lambda = np$ is "moderate"
- Different interpretations of "moderate"
  - $n > 20$ and $p < 0.05$
  - $n > 100$ and $p < 0.1$

- Really, Poisson is Binomial as

$$n \rightarrow \infty \text{ and } p \rightarrow 0, \text{ where } np = \lambda$$

Bin(10,0.3) vs Bin(100,0.03) vs Poi(3)



Poisson can be used to approximate a Binomial where n is large and p is small.

Tender (Central) Moments with Poisson

- Recall: Y ~ Bin(n, p)
  - E[Y] = np
  - Var(Y) = $np(1 - p)$
- X ~ Poi(λ) where $\lambda = np$ ($n \to \infty$ and $p \to 0$)
  - E[X] = $np = \lambda$
  - Var(X) = $np(1 - p) = \lambda(1 - 0) = \lambda$
  - Yes, expectation and variance of Poisson are same
    - It brings a tear to my eye…

A Real License Plate Seen at Stanford

No, it's not mine…
but I kind of wish it was.

Poisson is Chill

- Poisson can still provide a good approximation even when assumptions are “mildly” violated
- “Poisson Paradigm”
- Can apply Poisson approximation when...
    - “Successes” in trials are not entirely independent
        - Example: # entries in each bucket in large hash table
    - Probability of “Success” in each trial varies (slightly)
        - Small relative change in a very small p
        - Example: average # requests to web server/sec. may fluctuate slightly due to load on network

Web Server Load

- Consider requests to a web server in 1 second
  - In past, server load averages 2 hits/second
  - X = # hits server receives in a second
  - What is $P(X < 5)$?
- Solution

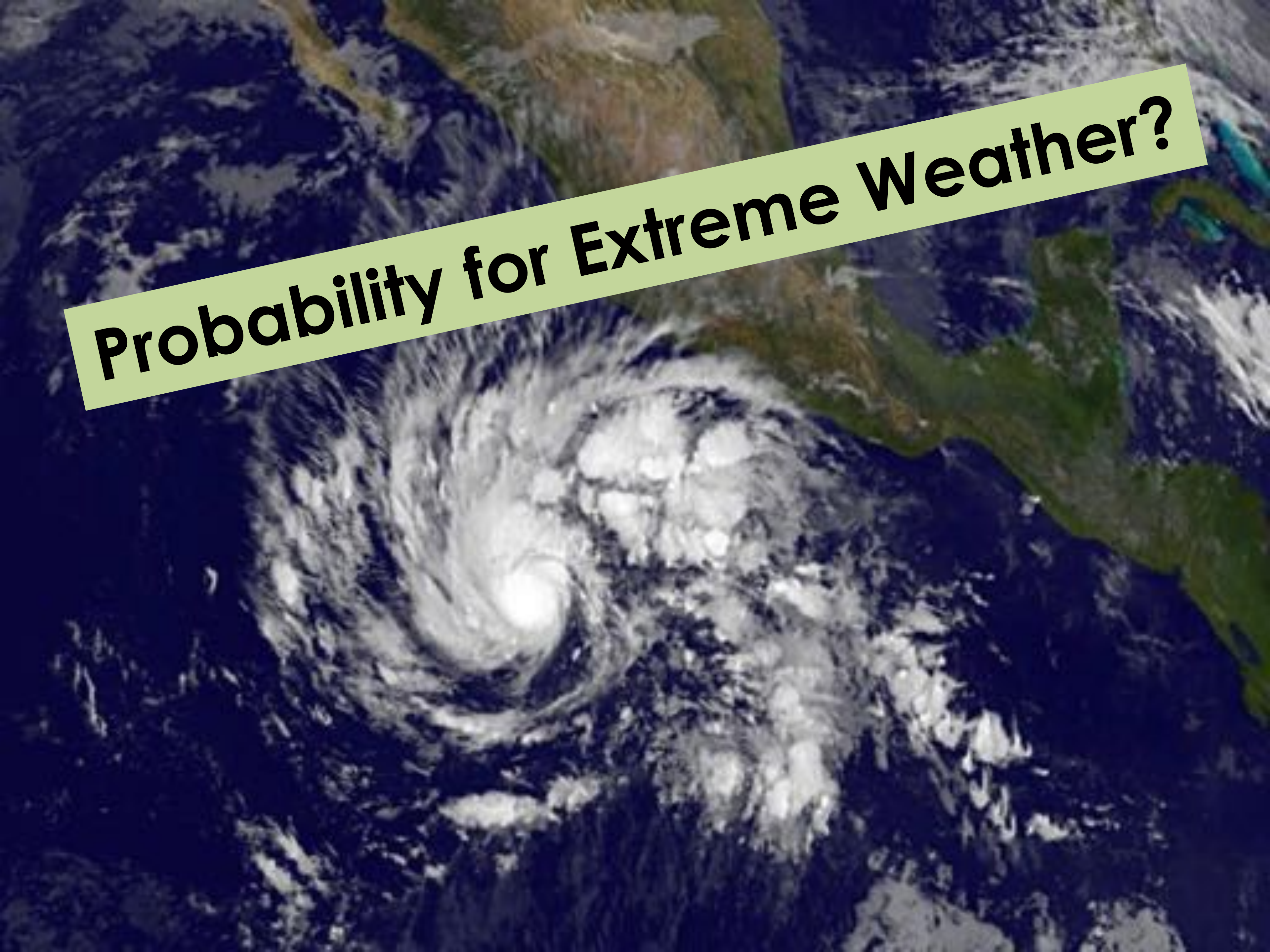
$$X \sim \text{Poi}(\lambda = 2)$$

$$P(X < 5) = \sum_{i=0}^{4} P(X = i)$$

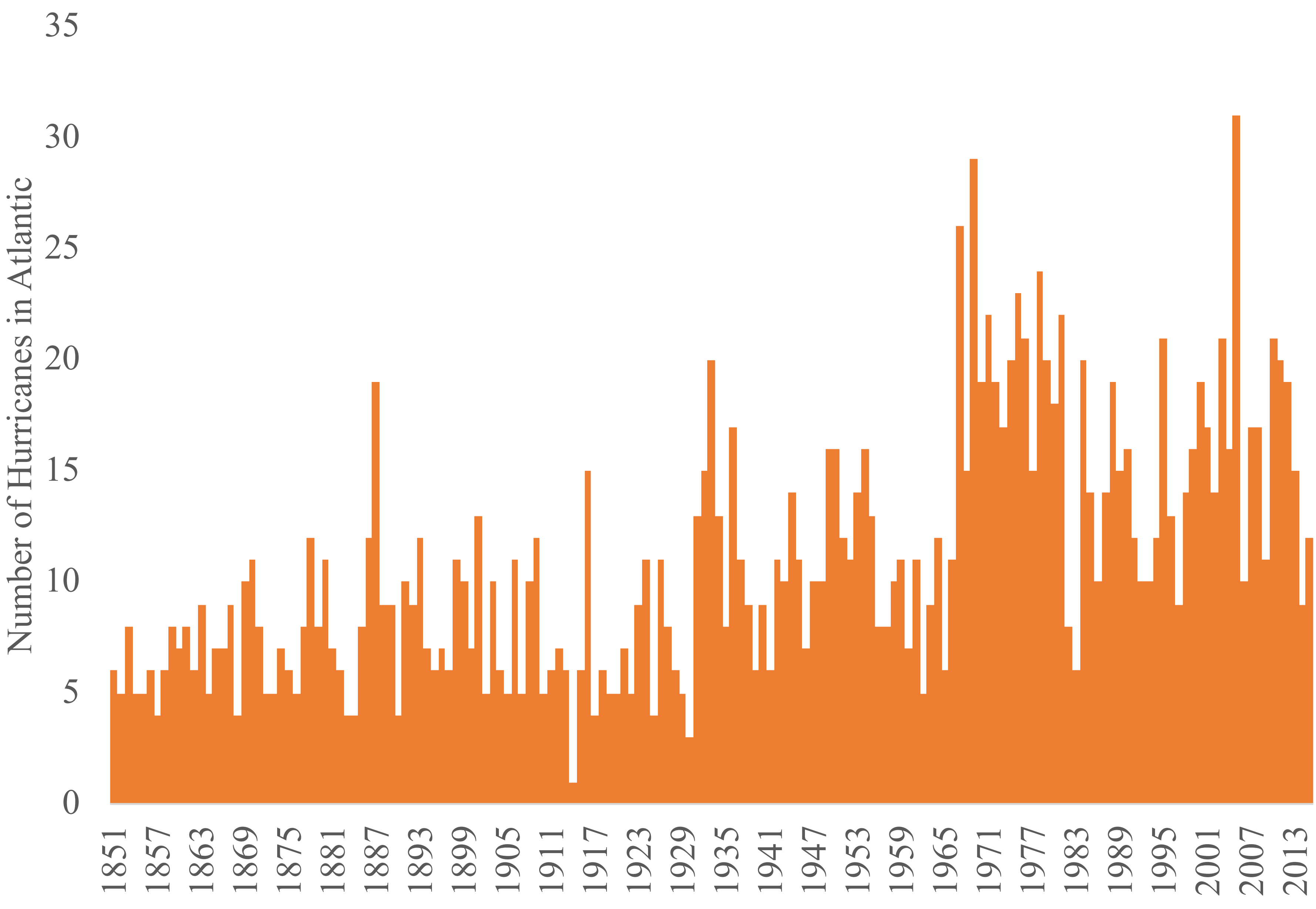
$$= \sum_{i=0}^{4} e^{-\lambda} \frac{\lambda^i}{i!} \qquad \text{Since } X \text{ is Poisson}$$

$$= \sum_{i=0}^{4} e^{-2} \frac{2^i}{i!} \approx 0.95 \qquad \text{Since } \lambda = 2$$

Probability for Extreme Weather?

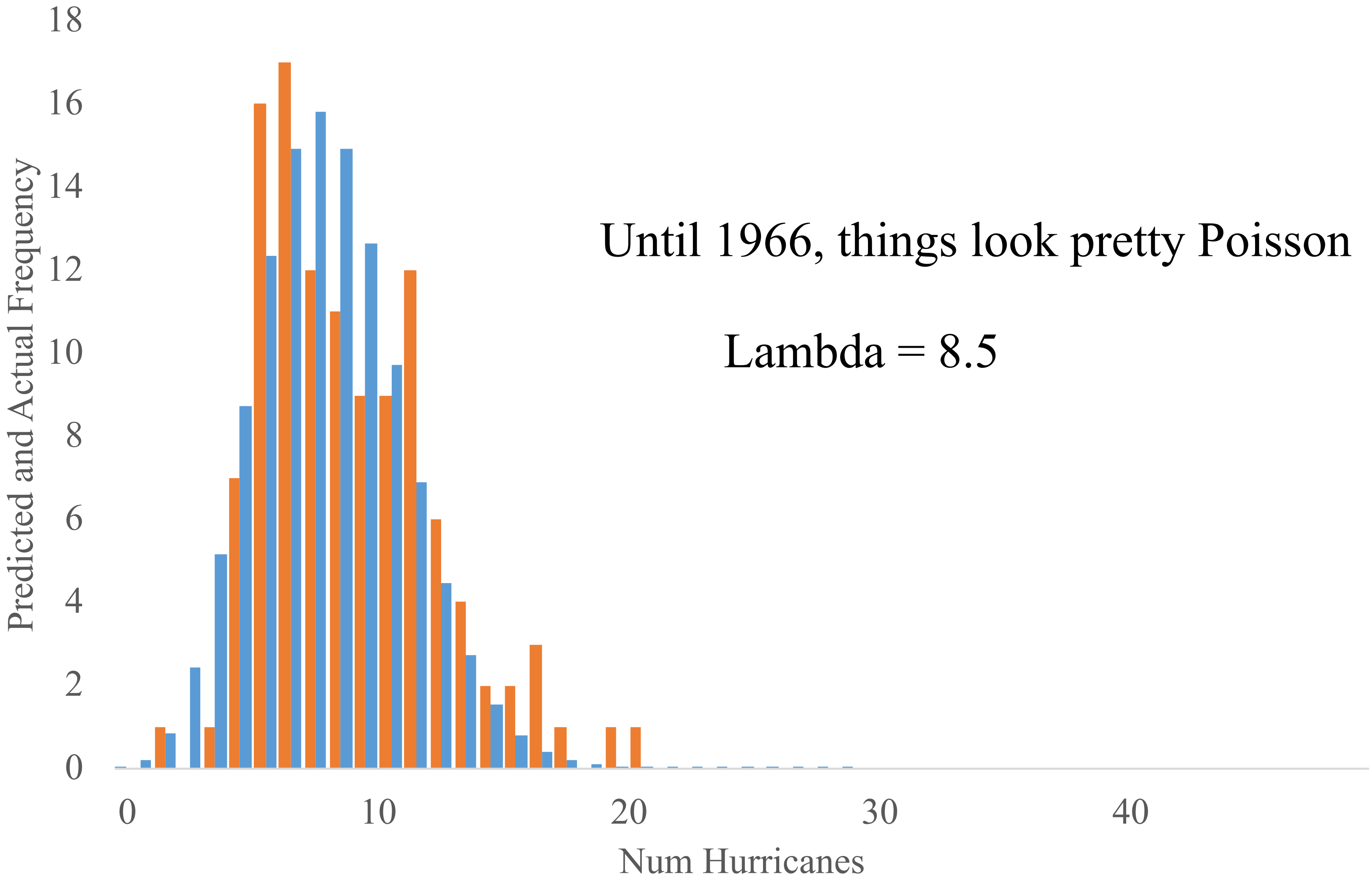


Hurricanes per Year since 1851



To the code!

Historically ~ Poisson(8.5)



Improbability Drive

- What is the probability of over 15 hurricanes in a season given that the distribution doesn't change?
  - Let X = # hurricanes in a year. X ~ Poi(8.5)
- Solution:

$$P(X > 15) = 1 - P(X \leq 15)$$

$$= 1 - \sum_{i=0}^{15} P(X = i)$$

$$= 1 - 0.98$$

$$= 0.02$$

This is the pmf of a Poisson. Your favorite programming language has a function for it

Twice since 1966 there have been years with over 30 hurricanes

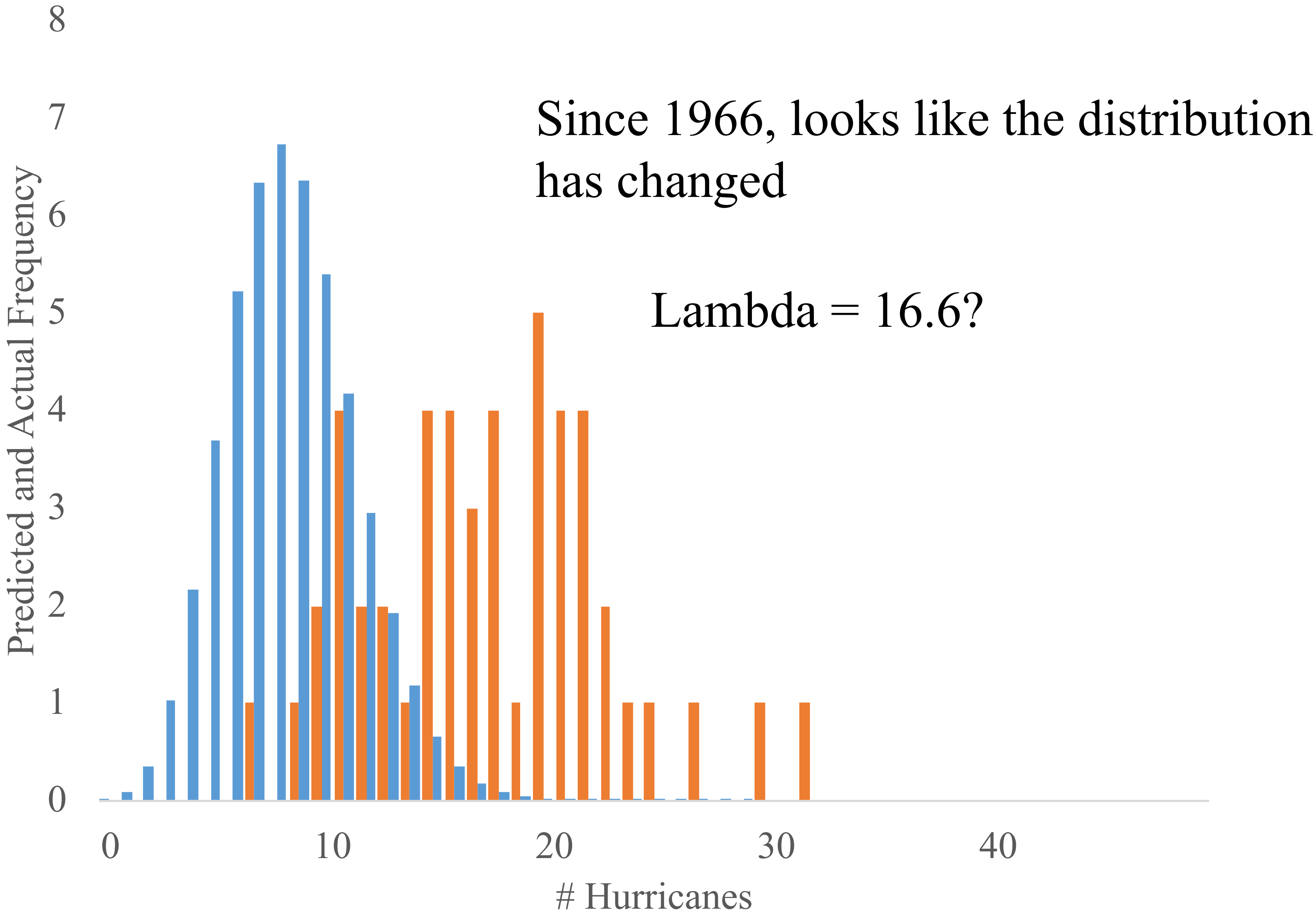
Improbability Drive

- What is the probability of over 30 hurricanes in a season given that the distribution doesn't change?
  - Let X = # hurricanes in a year. X ~ Poi(8.5)
- Solution:

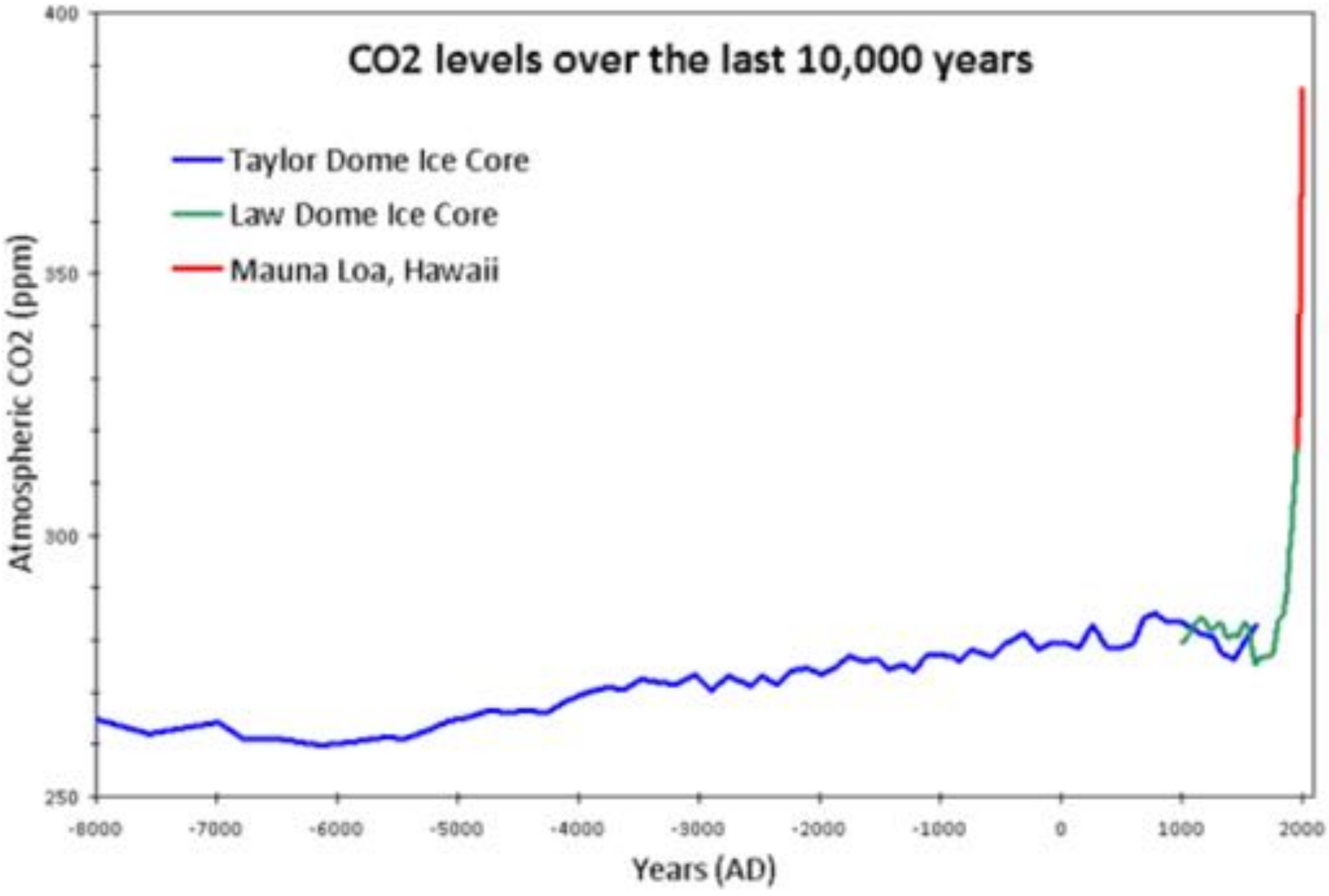
$$
\begin{aligned}
P(X > 30) &= 1 - P(X \leq 30) \\
&= 1 - \sum_{i=0}^{30} P(X = i) \\
&= 1 - 0.999999997823 \\
&= 2.2e - 09
\end{aligned}
$$

This is the pdf of a Poisson. Your favorite programming language has a function for it

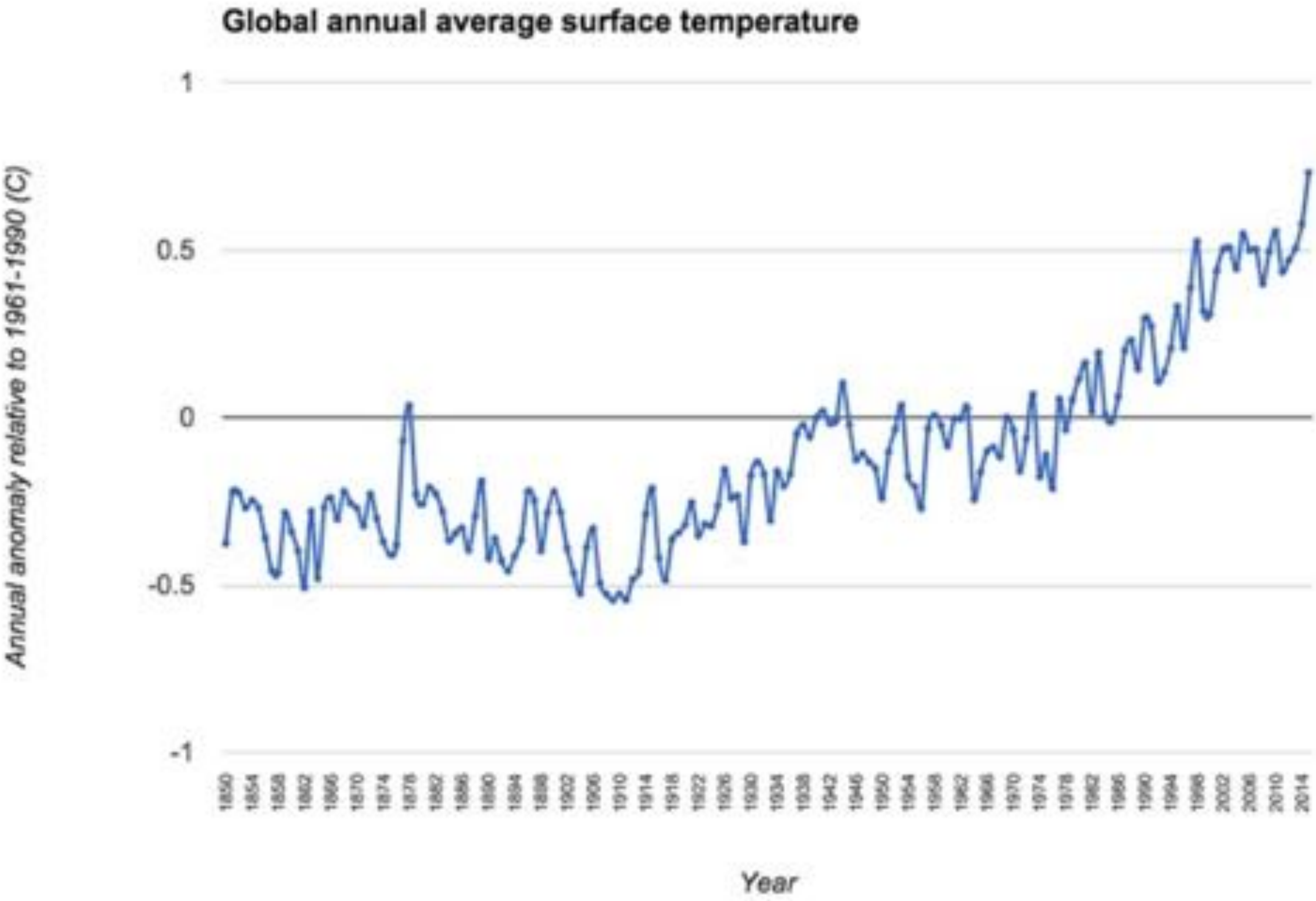
The Distribution has Changed



What's Up?


CO2 levels over the last 10,000 years
Taylor Dome Ice Core
Law Dome Ice Core
Mauna Loa, Hawaii
Atmospheric CO2 (ppm)
400
350
300
250
-8000
-7000
-6000
-5000
-4000
-3000
-2000
-1000
0
1000
2000
Years (AD)

What's Up?



Global annual average surface temperature
Annual anomaly relative to 1961-1990 (C)
1
0.5
0
-0.5
-1
1850
1854
1858
1862
1866
1870
1874
1878
1882
1886
1890
1894
1898
1902
1906
1910
1914
1918
1922
1926
1930
1934
1938
1942
1946
1950
1954
1958
1962
1966
1970
1974
1978
1982
1986
1990
1994
1998
2002
2006
2010
2014
Year

What's Up?

Python Scipy RV Methods

```
from scipy import stats # great package
X = stats.poisson(2.5)  # X ~ Poi(λ = 2.5)
print(X.pmf(2))         # P(X = 2)
```

Function	Description
`X.pmf(k)`	$P(X = k)$
`X.cdf(k)`	$P(X \leq k)$
`X.entropy()`	(Differential) entropy of X
`X.mean()`	$E[X]$
`X.var()`	$\text{Var}(X)$
`X,std()`	$\text{Std}(X)$

Next Time

Discrete Distributions

Bernoulli:

- indicator of coin flip X ~ Ber(*p*)

Binomial:

- # successes in *n* coin flips X ~ Bin(*n*, *p*)

Poisson:

- # successes in *n* coin flips X ~ Poi(λ)

Geometric:

- # coin flips until success X ~ Geo(*p*)

Negative Binomial:

- # trials until *r* successes X ~ NegBin(*r*, *p*)

Zipf:

- The popularity rank of a random word, from a natural language
- X ~ Zipf(s)

The Poisson Common Path